# Exact $N$-soliton solutions of the extended nonlinear Schrödinger equation

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By use of Hirota's direct method and a simple transformation, we obtain the exact $N$-soliton solutions of the extended nonlinear Schrödinger equation,

$$i\frac{\partial q}{\partial z}-\frac{k''}{2}\frac{\partial^2 q}{\partial t^2}+\beta|q|^2q+i\gamma\frac{\partial(|q|^2q)}{\partial t}-i\frac{k'''}{6}\frac{\partial^3 q}{\partial t^3}=0\,,$$

under the conditions $3k''\gamma=\beta k'''$ and $k''\gamma=\beta k'''$, respectively. The features of the solutions are discussed.

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## I. INTRODUCTION

It is well known that the propagation of a picosecond optical pulse in a monomode optical fiber (not including optical fiber loss) is described by the nonlinear Schrödinger (NLS) equation [1]

$$i\frac{\partial q}{\partial z}-\frac{k''}{2}\frac{\partial^2 q}{\partial t^2}+\beta|q|^2q=0\,. \tag{1}$$

In this equation, $q$ is a complex envelope amplitude, $t$ represents the time (in the group-velocity frame), $z$ represents the distance along the direction of propagation, $k''$ is the second derivative of the axial wave number $k(=2\pi/\lambda)$ with respect to the angular frequency $\omega$ of the light wave at the central frequency $\omega_0$ and describes the group-velocity dispersion, $\beta=n_2\omega_0/(cA_{\rm eff})$ is the effective nonlinear coefficient where $n_2$ is the Kerr coefficient of glass, $c$ is the speed of light in vacuum, and $A_{\rm eff}$ is the effective core area of the fiber. Hasegawa and Tappert [1] showed theoretically that an optical pulse in a dielectric fiber forms a solitary wave based on the fact that the wave envelope satisfies the NLS equation. Seven years after the prediction of Hasegawa and Tappert, Mollenauer, Stolen, and Gordon [2] succeeded in the generation and transmission of optical solitons in a fiber. However, since then further experimental and theoretical works [3,4] have shown that for femtosecond optical pulses, the NLS equation is no longer valid and the effect of high-order terms should be taken into account, and propagation of a femtosecond optical pulse in monomode optical fiber (not including linear and nonlinear optical fiber loss) can be described by the extended nonlinear Schrödinger equation

$$i\frac{\partial q}{\partial z}-\frac{k''}{2}\frac{\partial^2 q}{\partial t^2}+\beta|q|^2q+i\gamma\frac{\partial}{\partial t}(|q|^2q)-i\frac{k'''}{6}\frac{\partial^3 q}{\partial t^3}=0\,, \tag{2}$$

where $\gamma=2\beta/\omega_0$, $k'''$ is the third derivative of the axial wave number $k$ with respect to the angular frequency $\omega$ at $\omega=\omega_0$ and describes third-order dispersion. We have obtained the exact $N$-soliton solutions of Eq. (2) in the case of $k'''=0$ [5]. In general, Eq. (2) may not be completely integral. If $k''$, $\beta$, $\gamma$, and $k'''$ take some special value, the exact soliton solutions of Eq. (2) can exist. In the present paper, we give the exact $N$-soliton solutions of Eq. (2) in the cases $3k''\gamma=\beta k'''$ and $k''\gamma=\beta k'''$ by use of Hirota's direct method [6,7].

## II. EXACT $N$-SOLITON SOLUTIONS UNDER THE CONDITION $3k''=\beta k'''$

By making the transformation

$$q(z,t)=\frac{g(z,t)}{f(z,t)}\,, \tag{3}$$

and substituting Eq. (3) into (2), under the condition $3k''\gamma=\beta k'''$ we can obtain

$$\left[iD_z-\frac{k''}{2}D_t^2-\frac{ik'''}{6}D_t^3\right](gf)=0\,, \tag{4}$$

$$-\frac{k''}{2}D_t^2(ff)=\beta|g|^2\,, \tag{5}$$

$$D_t(g^*g)=0\,, \tag{6}$$

where $g(z,t)$ is a complex function and $f(z,t)$ is a real function with respect to $z$ and $t$, and the bilinear operator $D_z^mD_t^n$ is defined by

$$D_z^mD_t^n(gf)=\left[\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right]^m\left[\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right]^n g(z,t)\times f(z',t')|_{z'=z,t'=t}\,. \tag{7}$$

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The exact solutions of Eqs. (5)–(7) can be expressed as in the following forms

$$f(z,t)=\sum_{\mu=0,1}{}' \exp\left[\sum_{i,j(i<j)}^{(2N)} \varphi_{ij}\mu_i\mu_j+\sum_{i=1}^{2N}\mu_i\zeta_i\right], \quad (8)$$

$$g(z,t)=\sum_{\nu=0,1}{}'' \exp\left[\sum_{i,j(i<j)}^{(2N)} \varphi_{ij}\nu_i\nu_j+\sum_{i=1}^{2N}\nu_i\zeta_i\right], \quad (9)$$

$$g^*(z,t)=\sum_{\nu=0,1}{}''' \exp\left[\sum_{i,j(i<j)}^{(2N)} \varphi_{ij}\nu_i\nu_j+\sum_{i=1}^{2N}\nu_i\zeta_i\right], \quad (10)$$

with

$$\zeta_j=K_jz+\eta_jt+\zeta_j^0,\quad K_j=-i\frac{k''}{2}\eta_j^2+\frac{k'''}{6}\eta_j^3,$$
$$\zeta_{j+N}=\zeta_j^*,\quad K_{j+N}=K_j^*,\quad \eta_{j+N}=\eta_j \quad \text{for } j=1,2,\dots,N, \quad (11)$$

$$\varphi_{ij}=\ln\frac{\beta}{-k''(\eta_i+\eta_j)^2} \quad \text{for } i=1,2,\dots,N \text{ and } j=N+1,N+2,\dots,2N, \quad (12)$$

$$\varphi_{ij}=\ln\frac{-k''(\eta_i-\eta_j)^2}{\beta} \quad \text{for } i=1,2,\dots,N \text{ and } j=1,2,\dots,N \text{ or } i=N+1,N+2,\dots,2N \text{ and } j=N+1,N+2,\dots,2N, \quad (13)$$

where $*$ implies a complex conjugate, $\eta_i$ is a real parameter, $\zeta_i^0$ is a complex constant; $\sum'_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1=0,1$, $\mu_2=0,1,\dots,$ $\mu_{2N}=0,1$ under the condition $\sum_{i=1}^{N}\mu_i=\sum_{i=1}^{N}\mu_{i+N}$; $\sum''_{\nu=0,1}$ and $\sum'''_{\nu=0,1}$ indicate the summation over all possible combinations of $\nu_1=0,1$, $\nu_2=0,1,\dots,$ $\nu_{2N}=0,1$ under the conditions $\sum_{i=1}^{N}\nu_i=1+\sum_{i=1}^{N}\nu_{i+N}$ and $1+\sum_{i=1}^{N}\nu_i=\sum_{i=1}^{N}\nu_{i+N}$, respectively; and $\sum_{i,j>i=1}^{(2N)}$ indicates the summation over all possible pairs taken from $2N$ elements with the specified condition $j>i$, as indicated. We assume all $\eta_i$ are different from each other.

Then, we show that $f$ and $g$ defined by Eqs. (8)–(13) satisfy Eqs. (4)–(6). Substituting the expressions for $f$ and $g$ into Eqs. (4)–(6), we have

$$\sum_{\nu=0,1}{}''\sum_{\mu=0,1}{}'\left[i\sum_{i=1}^{2N}K_i\sigma_i-\frac{k''}{2}\left[\sum_{i=1}^{2N}\eta_i\sigma_i\right]^2-\frac{ik'''}{6}\left[\sum_{i=1}^{2N}\eta_i\sigma_i\right]^3\right]\exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}(\nu_i\nu_j+\mu_i\mu_j)+\sum_{i=1}^{2N}(\nu_i+\mu_i)\zeta_i\right]=0, \quad (14)$$

$$\frac{k''}{2}\sum_{\mu=0,1}{}'\sum_{\mu'=0,1}{}'\left[\sum_{i=1}^{2N}\eta_i\sigma'_i\right]^2\exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}(\mu_i\mu_j+\mu'_i\mu'_j)+\sum_{i=1}^{2N}(\mu_i+\mu'_i)\zeta_i\right]$$
$$+\beta\sum_{\nu'=0,1}{}'''\sum_{\nu=0,1}{}''\exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}(\nu'_i\nu'_j+\nu_i\nu_j)+\sum_{i=1}^{2N}(\nu'_i+\nu_i)\zeta_i\right]=0, \quad (15)$$

and

$$\sum_{\nu'=0,1}{}'''\sum_{\nu=0,1}{}''\left[\sum_{i=1}^{2N}\eta_i\sigma''_i\right]\exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}(\nu'_i\nu'_j+\nu_i\nu_j)+\sum_{i=1}^{2N}(\nu'_i+\nu_i)\zeta_i\right]=0, \quad (16)$$

where $\sigma_i=\nu_i-\mu_i$, $\sigma'_i=\mu_i-\mu'_i$, $\sigma''_i=\nu'_i-\nu_i$, for $i=1,2,\dots,2N$.

Let the coefficients of the factor

$$\exp\left[\sum_{i=1}^{L}\zeta_i+\sum_{i=1}^{L'}\zeta_{i+N}+\sum_{i=L+1}^{L+M}2\zeta_i+\sum_{i=L'+1}^{L'+M'}2\zeta_{i+N}\right],$$

in Eqs. (14)–(16) be $D_1$, $D_2$, and $D_3$, respectively. From Eq. (14), we have

$$D_1=\sum_{\nu=0,1}{}''\sum_{\mu=0,1}{}'C_{\nu\mu}\left[i\sum_{i=1}^{2N}K_i\sigma_i-\frac{k''}{2}\left[\sum_{i=1}^{2N}\eta_i\sigma_i\right]^2-\frac{ik'''}{6}\left[\sum_{i=1}^{2N}\eta_i\sigma_i\right]^3\right]\times\exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}(\nu_i\nu_j+\mu_i\mu_j)\right], \quad (17)$$

where $c_{\nu\mu}$ represents that the summation over $\nu$ and $\mu$ should be performed under the following conditions

$$\nu_i+\mu_i=1 \quad \text{for } i=1,2,\dots,L \text{ or } i-N=1,2,\dots,L',$$
$$\nu_i=\mu_i=1 \quad \text{for } i=L+1,L+2,\dots,L+M \text{ or } i-N=L'+1,L'+2,\dots,L'+M',$$
$$\nu_i=\mu_i=0 \quad \text{for } i=L+M+1,L+M+2,\dots,N \text{ or } i-N=L'+M'+1,L'+M'+2,\dots,N.$$

For convenience, let

$$\hat\sigma_i=\sigma_i,\quad \hat K_i=K_i,\quad \hat\eta_i=\eta_i \quad \text{for } i=1,2,\dots,L,$$
$$\hat\sigma_{i+L}=-\sigma_{i+N}\quad \hat K_{i+L}=-K_{i+N},\quad \hat\eta_{i+L}=-\eta_{i+N} \quad \text{for } i=1,2,\dots,L'.$$

Under the conditions above we find that the conditions

$$\sum_{i=1}^{N}\mu_i=\sum_{i=1}^{N}\mu_{i+N}\quad\text{and}\quad\sum_{\nu=1}^{N}\nu_i=1+\sum_{i=1}^{N}\nu_{i+N}$$

are compatible and each of them can be converted to

$$\sum_{i=1}^{L+L'}\hat{\sigma}_i=1$$

if, and only if $L+2M=1+L'+2M'$. Hence, we have, for $L+L'$=odd

$$D_1=A\hat{D}_1 = A\sum_{\hat{\sigma}=\pm 1}'\left[i\sum_{i=1}^{L+L'}\hat{K}_i\hat{\sigma}_i-\frac{k''}{2}\left[\sum_{i=1}^{L+L'}\hat{\eta}_i\hat{\sigma}_i\right]^2-\frac{ik'''}{6}\left[\sum_{i=1}^{L+L'}\hat{\eta}_i\hat{\sigma}_i\right]^3\right]\times\prod_{i,j(i<j)}^{(L+L')}\left[\frac{-k''(\hat{\eta}_i-\hat{\eta}_j)^2}{\beta}\right]^{1/2(1+\hat{\sigma}_i\hat{\sigma}_j)}, \tag{18}$$

where $A$ is independent on $\hat{\sigma}$, $\sum'_{\hat{\sigma}=\pm1}$ implies the summation over all possible combinations of $\hat{\sigma}_1=\pm1, \hat{\sigma}_2=\pm1,\ldots,\hat{\sigma}_{L+L'}=\pm1$ under the condition $\sum_{i=1}^{L+L'}\hat{\sigma}_i=1$ and $\prod_{i,j(i<j)}^{(L)}$ indicates the product of *all* possible combinations of pairs chosen from $L+L'$ elements with the specified condition $j>i$.

For $L+L'$=even, similar procedures give

$$\hat{D}_2=\left[\frac{k''}{2}\sum_{\hat{\sigma}=\pm1}''\left[\sum_{i=1}^{L+L'}\hat{\eta}_i\hat{\sigma}_i\right]^2+\beta\sum_{\hat{\sigma}=\pm1}'''\right]\times\prod_{i,j(i<j)}^{(L+L')}\left[\frac{-k''(\hat{\eta}_i-\hat{\eta}_j)^2}{\beta}\right]^{1/2(1+\hat{\sigma}_i\hat{\sigma}_j)}, \tag{19}$$

and

$$\hat{D}_3=\sum_{\hat{\sigma}=\pm1}'''\left[\sum_{i=1}^{L+L'}\hat{\eta}_i\hat{\sigma}_i\right]\prod_{i,j(i<j)}^{(L+L')}\times\left[\frac{-k''(\hat{\eta}_i-\hat{\eta}_j)^2}{\beta}\right]^{1/2(1+\hat{\sigma}_i\hat{\sigma}_j)}, \tag{20}$$

where $\sum''_{\hat{\sigma}_i=\pm1}$ and $\sum'''_{\hat{\sigma}=\pm1}$ imply the summation over all possible combinations of $\hat{\sigma}_1=\pm1$, $\hat{\sigma}_2=\pm1,\ldots,\hat{\sigma}_{L+L'}=\pm1$ under the conditions $\sum_{i=1}^{L+L'}\hat{\sigma}_i=0$ and $\sum_{i=1}^{L+L'}\hat{\sigma}_i=-2$, respectively. Thus, $f$ and $g$ defined by Eqs. (8)–(13) are the solutions of Eqs. (4)–(6) provided that the following identities hold:

$$\hat{D}_1=0 \quad\text{for odd } n=L+L', \tag{21}$$

$$\hat{D}_2=0 \text{ and } \hat{D}_3=0 \quad\text{for even } n=L+L'. \tag{22}$$

We shall prove the identities by the following method. $\hat{D}_1$, $\hat{D}_2$, and $\hat{D}_3$ have the following properties: (i) $\hat{D}_1$, $\hat{D}_2$, and $\hat{D}_3$ are symmetric polynomials of $\eta_1, \eta_2, \ldots, \eta_n$: (ii) if $\eta_1=\eta_2$, then

$$\hat{D}_1(\hat{\eta}_1,\hat{\eta}_2,\ldots,\hat{\eta}_n)=2\prod_{j=3}^{n}\frac{-k''(\hat{\eta}_1-\hat{\eta}_j)^2}{\beta}\hat{D}_1(\hat{\eta}_3,\hat{\eta}_4,\ldots,\hat{\eta}_n), \tag{23}$$

$$\hat{D}_2(\hat{\eta}_1,\hat{\eta}_2,\ldots,\hat{\eta}_n)=2\prod_{j=3}^{n}\frac{-k''(\hat{\eta}_1-\hat{\eta}_j)^2}{\beta}\hat{D}_2(\hat{\eta}_3,\hat{\eta}_4,\ldots,\hat{\eta}_n), \tag{24}$$

$$\hat{D}_3(\hat{\eta}_1,\hat{\eta}_2,\ldots,\hat{\eta}_n)=2\prod_{j=3}^{n}\frac{-k''(\hat{\eta}_2-\hat{\eta}_j)^2}{\beta}\hat{D}_3(\hat{\eta}_3,\hat{\eta}_4,\ldots,\hat{\eta}_n). \tag{25}$$

The identities $\hat{D}_1=0$, $\hat{D}_2=0$, and $\hat{D}_3=0$ are easily verified for $n=1$ and $n=2$, respectively. Now, we assume the identities hold for $n-2$. Then, relying on the properties (i) and (ii), it is seen that $\hat{D}_1$, $\hat{D}_2$, and $\hat{D}_3$ can be, respectively, factorized by a symmetric polynomial

$$\prod_{i,j(i<j)}^{(n)}(\hat{\eta}_i-\hat{\eta}_j)^2$$

of degree $n(n-1)$. On the other hand, Eqs. (18)–(20) show that the degree of $\hat{D}_1$, $\hat{D}_2$, and $\hat{D}_3$ are $(n-1)^2/2+3$, $n(n-1)/2+2$, and $n(n-1)/2+3$, respectively. Hence, $\hat{D}_1$, $\hat{D}_2$, and $\hat{D}_3$ must be zero for $n$, and the identities have been proved.

From Eqs. (8)–(13), we can obtain a fundamental soliton solution of Eq. (2)

$$q(z,t)=\left[-\frac{k''}{\beta}\right]^{1/2}\eta\,\text{sech}\times\left[\eta\left[t+\frac{k'''}{6}\eta^2 z\right]\right]e^{-ik''\eta^2 z/2}, \tag{26}$$

where $\eta$ is the inverse of the soliton width, and forms of $f$ and $g$ for $N=2$

$$f=1-\frac{\beta e^{\zeta_1+\zeta_1^*}}{4k''\eta_1^2}-\frac{\beta(e^{\zeta_1+\zeta_2^*}+e^{\zeta_1^*+\zeta_2})}{k''(\eta_1+\eta_2)^2}-\frac{\beta e^{\zeta_2+\zeta_2^*}}{4k''\eta_2^2}+\frac{\beta^2(\eta_1-\eta_2)^4 e^{\zeta_1+\zeta_2+\zeta_1^*+\zeta_2^*}}{16(k'')^2\eta_1^2\eta_2^2(\eta_1+\eta_2)^4}, \tag{27}$$

$$g=e^{\zeta_1}+e^{\zeta_2}-\frac{\beta(\eta_1-\eta_2)^2 e^{\zeta_1+\zeta_2+\zeta_1^*}}{4k''\eta_1^2(\eta_2+\eta_1)^2}-\frac{\beta(\eta_1-\eta_2)^2 e^{\zeta_1+\zeta_2+\zeta_2^*}}{4k''\eta_2^2(\eta_1+\eta_2)^2}. \tag{28}$$

Taking

$$\zeta_1^0=-\tfrac{1}{2}\ln\frac{\beta(\eta_1-\eta_2)^2}{-4k''\eta_1^2(\eta_1+\eta_2)^2},$$

$$\zeta_2^0=-\tfrac{1}{2}\ln\frac{\beta(\eta_1-\eta_2)^2}{-4k''\eta_2^2(\eta_1+\eta_2)^2},$$

in Eqs. (27) and (28), we obtain the two-soliton solution of

Eq. (2)

$$q(z,t)=\left[\frac{-k''}{\beta}\right]^{1/2}\frac{2(\eta_1+\eta_2)}{|\eta_1-\eta_2|}\left[\eta_1\cosh\left[\eta_2 t+\frac{k'''}{6}\eta_2^3 z\right]\exp\left[-i\frac{k''}{2}\eta_1^2 z\right]+\eta_2\cosh\left[\eta_1 t+\frac{k'''}{6}\eta_1^3 z\right]\exp\left[-i\frac{k''}{2}\eta_2^2 z\right]\right]$$
$$\times\left\{\cosh\left[(\eta_1+\eta_2)t+\frac{k'''}{6}(\eta_1^3+\eta_2^3)z\right]+\frac{(\eta_1+\eta_2)^2}{(\eta_1-\eta_2)^2}\cosh\left[(\eta_1-\eta_2)t+\frac{k'''}{6}(\eta_1^3-\eta_2^3)z\right]\right.$$
$$\left.+\frac{4\eta_1\eta_2}{(\eta_1-\eta_2)^2}\cos\left[-\frac{k''}{2}(\eta_1^2-\eta_2^2)z\right]\right\}^{-1}, \tag{29}$$

having taken $k'''=\gamma=0$, $\eta_1=1$, and $\eta_2=3$, the soliton solutions (26) and (29) become those of the NLS equation, respectively [3].

It is easily seen that the solutions (26) and (29) are different from those of the NLS equation only in the following aspects: (i) The velocity of the soliton (in group-velocity frame) is proportional to third-order dispersion and square of its width (or the inverse of its amplitude), the soliton of the taller peak travels slower and is narrower; (ii) Two-soliton solution (29) is the nonlinear overlap of two solitons with different amplitude (or width) moving along propagation direction at different velocities, which results in breakup of the bound state of the soliton and evolves into two completely separated solitons. Therefore, the solution (29) is aperiodic and third-order dispersion acts to decompose the bond soliton into its component solitons. The results above are in agreement with the analysis in Ref. [4]. It can be shown that the conclusions above are valid for the higher-order solitons.

## III. EXACT N-SOLITON SOLUTIONS UNDER THE CONDITION $k''\gamma=\beta k'''$

In order to obtain the soliton solutions of Eq. (2) under the condition $k''\gamma=\beta k'''$, letting

$$q(z,t)=\rho(z,T)\exp[i(\Delta\omega T-\Delta k z)], \tag{30}$$

with

$$\Delta\omega=\frac{k''}{k'''},\quad T=t+\frac{(k'')^2}{2k'''}z,\quad \Delta k=\frac{(k'')^3}{6(k''')^2}, \tag{31}$$

and substituting Eq. (30) into (2), under the condition $k''\gamma=\beta k'''$ we obtain the envelope equation corresponding to Eq. (2)

$$\frac{\partial\rho}{\partial z}-\frac{k'''}{6}\frac{\partial^3\rho}{\partial T^3}+3\gamma\rho^2\frac{\partial\rho}{\partial T}=0, \tag{32}$$

where $T$ represents the time in a new group velocity frame, $\omega$ and $k$ correspond to the constant shifts relative to the original carrier frequency $\omega_0$ and wave number $k_0$, $\rho(z,T)$ is a real function of the $z$ and $T$ which is really the envelope function of light wave. By making the transformation

$$\rho(z,T)=\frac{g(z,T)}{f(z,T)}, \tag{33}$$

and substituting Eq. (33) into (32), we can obtain

$$\left[D_z-\frac{k'''}{6}D_T^3\right](gf)=0, \tag{34}$$

$$-\frac{k'''}{6}D_T^2(ff)=\gamma g^2, \tag{35}$$

where $g(z,T)$ and $f(z,T)$ are real functions of $z$ and $T$. The exact solutions of Eqs. (34) and (35) can be expressed as in the following forms

$$f(z,T)=\sum_{\mu=0,1}{}' \exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}\mu_i\mu_j+\sum_{i=1}^{2N}\mu_i\zeta_i\right], \tag{36}$$

$$g(z,T)=\sum_{\nu=0,1}{}'' \exp\left[\sum_{i,j(i<j)}^{(2N)}\varphi_{ij}\nu_i\nu_j+\sum_{i=1}^{2N}\nu_i\zeta_i\right], \tag{37}$$

with

$$\zeta_i=K_i z+\eta_i T+\zeta_i^0,\quad K_i=+\frac{k'''}{6}\eta_i^3,$$
$$\zeta_{i+N}=\zeta_i,\quad \eta_{i+N}=\eta_i,\quad K_{i+N}=K_i \quad \text{for } i=1,2,\ldots,N, \tag{38}$$

$$\varphi_{ij}=\ln\frac{3\gamma}{-k'''(\eta_i+\eta_j)^2}\quad \text{for } i=1,2,\ldots,N \text{ and } j=N+1,N+2,\ldots,2N, \tag{39}$$

$$\varphi_{ij}=\ln\frac{-k'''(\eta_i-\eta_j)^2}{3\gamma}\quad \text{for } i=1,2,\ldots,N \text{ and } j=1,2,\ldots,N, \text{ or } i=N+1,N+2,\ldots,2N \text{ and } j=N+1,N+2,\ldots,2N, \tag{40}$$

where $\eta_i$ and $\zeta_i^0$ are real parameters of the $i$th soliton; $\sum'_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1=0,1$, $\mu_2=0.1,\ldots$, $\mu_{2N}=0,1$ under the condition $\sum_{i=1}^{N}\mu_i=\sum_{i=1}^{N}\mu_{i+N}$; $\sum''_{\nu=0,1}$ indicates the summation over all possible combinations of $\nu_1=0,1$, $\nu_2=0,1,\ldots,\nu_{2N}=0,1$ under the conditions $\sum_{i=1}^{N}\nu_i=1+\sum_{i=1}^{N}\nu_{i+N}$; and $\sum_{i,j(i<j)}^{(2N)}$ indicates the summation over all possible pairs taken from $2N$ elements with the specified condition $j>i$, as indicated. We assume all $\eta_i$ are different from each other.

By the same method as used in Sec. II, it can be proved that $f$ and $g$ defined by Eqs. (36) and (37) satisfy Eqs. (34) and (35), which is no longer repeated here.

From Eqs. (30)–(40), we can obtain a fundamental soliton solution of Eq. (32)

$$\rho(z,T)=\left[\frac{-k'''}{3\gamma}\right]^{1/2}\eta\,\mathrm{sech}\left[\eta\left[T+\frac{k'''}{6}\eta^2 z\right]\right], \qquad (41)$$

where $\eta$ is the inverse of the soliton width, and forms of $f$ and $g$ for $N=2$

$$f=1-\frac{3\gamma e^{2\xi_1}}{4k'''\eta_1^2}-\frac{3\gamma e^{\xi_1+\xi_2}}{k'''(\eta_1+\eta_2)^2}-\frac{3\gamma e^{2\xi_2}}{4k'''\eta_2^2}+\frac{9\gamma^2(\eta_1-\eta_2)^4 e^{2\xi_1+2\xi_2}}{16(k''')^2\eta_1^2\eta_2^2(\eta_1+\eta_2)^4}, \qquad (42)$$

$$g=e^{\xi_1}+e^{\xi_2}-\frac{3\gamma(\eta_1-\eta_2)^2 e^{2\xi_1+\xi_2}}{k'''\eta_1^2(\eta_1+\eta_2)^2}-\frac{3\gamma(\eta_1-\eta_2)^2 e^{\xi_1+2\xi_2}}{k'''\eta_2^2(\eta_1+\eta_2)^2}. \qquad (43)$$

Taking

$$\xi_1^0=-\tfrac{1}{2}\ln\frac{3\gamma(\eta_1-\eta_2)^2}{-4k'''\eta_1^2(\eta_1+\eta_2)^2},$$

$$\xi_2^0=-\tfrac{1}{2}\ln\frac{3\gamma(\eta_1-\eta_2)^2}{-4k'''\eta_2^2(\eta_1+\eta_2)^2},$$

in Eqs. (42) and (43), we obtain the two soliton solution

$$\rho(z,T)=\sqrt{-k'''/3\gamma}\,\frac{2(\eta_1+\eta_2)}{|\eta_1-\eta_2|}\left[\eta_1\cosh\left[\eta_2 T+\frac{k'''}{6}\eta_2^3 z\right]+\eta_2\cosh\left[\eta_1 T+\frac{k'''}{6}\eta_1^3 z\right]\right]$$
$$\times\left\{\cosh\left[(\eta_1+\eta_2)T+\frac{k'''}{6}(\eta_1^3+\eta_2^3)z\right]+\frac{(\eta_1+\eta_2)^2}{(\eta_1-\eta_2)^2}\cosh\left[(\eta_1-\eta_2)T+\frac{k'''}{6}(\eta_1^3-\eta_2^3)z\right]+\frac{4\eta_1\eta_2}{(\eta_1-\eta_2)^2}\right\}^{-1}. \qquad (44)$$

The soliton solutions (41) and (44) are in agreement with those of the modified Korteweg–de Vries equation [6,7]. Having used transformation (31), we only can take $k''=\beta=0$, instead of $\gamma=K'''=0$; $k''$ only makes constant shifts to the carrier frequency $\omega_0$ and wave number $k_0$, when $k''=\beta=0$, Eq. (2) becomes (32). The velocity of the soliton (in the group-velocity frame) is proportional to third-order dispersion and square of its width (or the inverse of its amplitude), the higher-order solitons are aperiodic and third-order dispersion acts to decompose the bond soliton into its component solitons, which is in agreement with that of the solutions (26) and (29).

## IV. SUMMARY AND CONCLUSIONS

In this paper, we obtain the exact $N$-soliton solutions of the extended nonlinear Schrödinger equation under the conditions $3k''\gamma=\beta k'''$ and $k''\gamma=\beta k'''$, respectively. The soliton solutions under both of the above conditions show that the velocity of the soliton (in the group-velocity frame) is proportional to third-order dispersion and square of its width (or the inverse of its amplitude); the higher-order solitons are aperiodic and third-order dispersion acts to decompose the bond soliton into its component solitons.

It should be noted that under the condition $k''\gamma=\beta k'''$, the soliton solutions are the same as those of the modified Korteweg–de Vries equation. The modified Korteweg–de Vries equation is derived in the study of anharmonic lattices [6,8]. The above results show that under the appropriate condition, femtosecond optical soliton pulses propagating in a monomode fiber may have the same features as those of the solitons in anharmonic lattices, which is very important and interesting.


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